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Some non-perturbative calculations on spin glasses

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Abstract. Models of spin glasses are studied with a phase transition which is discontinuous in the Parisi order parameter. It is assumed that the leading-order corrections to the thermodynamic limit of the high-temperature free energy are due to the existence of a metastable saddle point in the replica formalism. An ansatz is made for the form of the metastable point and its contribution to the free energy is calculated. The random energy model is considered along with the Ising spin glass with p-spin interactions and the p-state Potts glass with two-spin interactions in their $p < \infty$ expansion.

1. Introduction

The mean-field replica theory has proved to be a very powerful method for studying spin-glass models [1]. The main problem with the replica method is that it necessarily requires an ansatz for the solution. The random energy model (REM) is a very simplified spin-glass model which can also be solved without the use of replicas. It is, therefore, a very good testing ground for the replica method since, although very simple, it presents the typical features of spin glasses such as replica symmetry breaking and non-ergodicity. For T greater than a critical temperature T_c the REM presents no breaking of the permutation symmetry among the replicas which, in the absence of external fields, tend to have the lowest possible value for the mean overlap. When $T < T_c$ the system undergoes a phase transition towards a one-replica-symmetry-broken phase. The corresponding latent heat is zero and so the transition is second order. The REM was first introduced and solved by Derrida [2] who was also able to calculate the finite-size corrections to the high-temperature free energy.

In the present work this last result will be obtained by making use of the replica method. In doing so, a metastable saddle point in the replica space will be individuated as responsible for the leading-order finite-size corrections. The Ising spin glass with p-spin interactions (p-spin model) and the p-state Potts model will also be considered. Both these models tend to a REM in the limit $p \to \infty$.

The scheme of this paper is as follows. Section 1 is based on reference [2]; the REM and Derrida's results for the finite-size corrections will be presented. In section 2 the same results as the contribution of a metastable saddle point will be recovered. Section 3 will introduce the p-spin model and repeat the calculation performed for the REM in the formalism of a ($p = \infty$)-spin model. In section 4 the results will be extended to the case $p < \infty$. In section 5 the p-state Potts model will be introduced and the $p = \infty$ limit will be studied. Finally, a $p < \infty$ expansion for the Potts model will be formulated and the calculations will be extended to this case.

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2. The random energy model

It is worthwhile recalling the main results from the REM in order to establish the notation. The model describes the behaviour of any system with a fixed number of energy levels with the energies independently distributed according to a Gaussian law. If 2^N is the number of levels, the model is defined by the properties:

$$P(E) = \frac{1}{\sqrt{N\pi J^2}} \exp\left[\frac{-E^2}{NJ^2}\right] \tag{1}$$

$$P(E_i, E_j) = P(E_i)P(E_j). \tag{2}$$

The solution can easily be derived using a microcanonical argument. For the free energy one finds

$$F = \begin{cases} N\left[-\frac{\ln 2}{\beta} - \frac{\beta J^2}{4}\right] & \text{for } \beta < \beta_c \quad (3a) \\ -E_0 & \text{for } \beta > \beta_c \quad (3b) \end{cases}$$

where $E_0 = NJ\sqrt{\ln 2}$ and $\beta_c = 2\sqrt{\ln 2}/J$.

The REM can also be solved by making use of replicas. If one computes the nth power of the partition function one obtains

$$Z^n = \sum_{\{p_i\}} \frac{n!}{\prod_i (p_i!)} \exp\left[-\sum_i^{2^N} \frac{p_i E_i}{T}\right] \tag{4}$$

where

$$p_i \geqslant 0 \qquad \sum_i^{2^N} p_i = n. \tag{5}$$

After averaging over the disorder one has

$$\overline{Z^n} = \sum_{\{\nu\}} \frac{n!}{\prod_{p=1}^n (p!)^{\nu_p!} \prod_{p=1}^n \nu_p!} \exp\left[N \sum_{p=0}^n \nu_p \left[\ln 2 + \frac{p^2\beta^2J^2}{4}\right]\right] \tag{6}$$

where ν_p is the number of p_i that are equal to p. The ν_p verify the conditions

$$\nu_p \geqslant 0 \qquad \sum_{p=0}^n \nu_p = 2^N \qquad \sum_{p=1}^n p\nu_p = n. \tag{7}$$

To obtain equation (6) we have used the fact that for large enough N we can write

$$\frac{(2^N)!}{\nu_0!} \sim 2^{N\sum_{p=1}^n \nu_p}. \tag{8}$$

One can find that for $T > \sqrt{n}T_c$ the dominant contribution of (6) is obtained by taking

$$\nu_1 = n \tag{9}$$

$$\nu_{p\geqslant 2} = 0. \tag{10}$$

The corresponding expression for $\overline{Z^n}$ is linear in n, so one can use the well known formula

$$\overline{\ln Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n} \tag{11}$$

to calculate the high-temperature free energy (3a). The low-temperature expression for (3) is obtained by taking $\nu_{p\neq\bar{p}} = 0$ and $\nu_{p=\bar{p}} = n/\bar{p}$, where $\bar{p}$ comes out to be equal to T/T_c.

Finally one can write, without deriving it, the expression for the finite-size high-temperature free energy:

$$\overline{\ln Z} = N\left[\ln 2 + \frac{J^2}{4T^2}\right] - \frac{1}{2}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^2} + \frac{1}{3}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^3} + \cdots + \frac{(-1)^{k+1}}{k}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^k}$$
$$+ \frac{2T\sqrt{\pi}}{J\left[\frac{T^2}{T_c^2}+1\right]\sqrt{N}\sin\left(\frac{\pi}{2}\left[\frac{T^2}{T_c^2}+1\right]\right)}\exp\left[-\frac{NT^2J^2}{16}\left[\frac{1}{T_c^2} - \frac{1}{T^2}\right]^2\right] + \cdots$$
$$\sqrt{2k-1}T_c < T < \sqrt{2k+1}T_c. \tag{12}$$

Derrida derives expression (12) for $k = 1, 2, 3$. He also makes the more general hypothesis that it should be true for every $k \geqslant 1$.

3. The metastable point

In this section the replica method will be used to derive equation (12). It has been asserted that, for $T > T_c$, the dominant contribution to the equation (6) is given by the choice (10). The effect of taking only one group of m replicas will now be considered. This can be done by taking

$$\nu_1 = n - m \qquad \nu_m = 1. \tag{13}$$

For large N, equation (6) can thus be written as follows

$$\overline{Z^n} = \overline{Z^n}_{\text{dom}} + \overline{Z^n}_{\text{sub}} \tag{14}$$

where dom stands for 'dominant' and sub stands for 'subdominant'. $\overline{Z^n}_{\text{dom}}$ is given by the choice (10) in equation (6) while $\overline{Z^n}_{\text{sub}}$ is given by the choice (13). One then has

$$\overline{\ln Z} = N\left[\ln 2 + \frac{J^2}{4T^2}\right] + \lim_{n\to 0}\frac{1}{n}\overline{Z^n}_{\text{sub}} \tag{15}$$

where in $\overline{Z^n}_{\text{sub}}$ all the integer m greater than $m = 1$ are summed over. One then has

$$\lim_{n\to 0}\frac{1}{n}\overline{Z^n}_{\text{sub}} = -\sum_{m=2}^{\infty}\frac{(-1)^m}{m}\exp N\left[(1-m)\ln 2 + \frac{\beta^2J^2}{4}(m^2-m)\right]. \tag{16}$$

The sum in (16) can be written as an integral in the complex plane over the circuit (C) as shown in figure 1.

One has

$$\sum_{m=2}^{\infty}\frac{(-1)^m}{m}\exp N\left[(1-m)\ln 2 + \frac{\beta^2J^2}{4}(m^2-m)\right]$$
$$= -\frac{1}{2}\int_C \mathrm{d}m\,\frac{\exp\left\{N\left[(1-m)\ln 2 + \frac{\beta^2J^2}{4}(m^2-m)\right]\right\}}{m\sin(\pi m)}. \tag{17}$$

Both the sum and the integral are not well defined. They can be defined by deforming the circuit C into a vertical path as indicated in figure 2. It can then be written

$$\sum_{m=2}^{\infty}\frac{(-1)^m}{m}\exp N\left[(1-m)\ln 2 + \frac{\beta^2J^2}{4}(m^2-m)\right]$$
$$\equiv -\frac{1}{2}\int_{\uparrow} \mathrm{d}m\,\frac{\exp\left\{N\left[(1-m)\ln 2 + \frac{\beta^2J^2}{4}(m^2-m)\right]\right\}}{m\sin(\pi m)} + \mathcal{R} \tag{18}$$

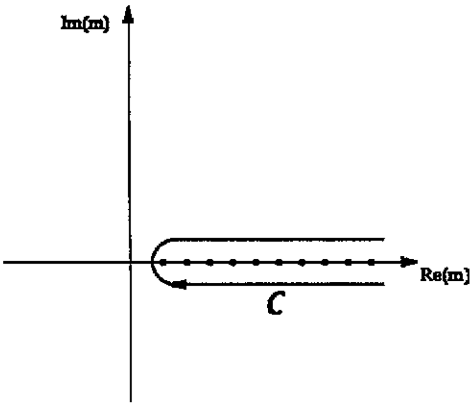

Figure 1. Integration path C.

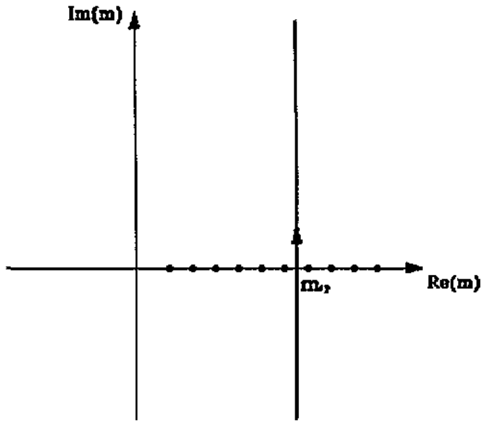

Figure 2. Integration path (↑). The path intersects the Re(m) axis at the point m_{sp}.

where $\mathcal{R}$ is the residual contribution mentioned below. The integral on the right-hand side of the previous equation can be solved with the aid of the saddle-point method. Two remarks have to be made on this subject. The first is that since the integration is over imaginary dm, and, since the exponent is quadratic in m, the saddle point m_{sp} is given by the minimum and not by the maximum in m. Therefore

$$m_{sp} = \frac{1}{2}\left(1 + \frac{T^2}{T_c^2}\right). \tag{19}$$

The second remark is that to the saddle-point contribution one has to add the terms of the sum in equation (16) such that $2 \leqslant m < m_{sp}$, where m_{sp} is no longer an integer. The evaluation of the integrand on m_{sp} gives exactly the last term in equation (12). The terms

$$\frac{(-1)^{k+1}}{k}\overline{\left[\frac{Z}{\overline{\overline{Z}}} - 1\right]^k}$$

are reproduced by the residues that are added. The number k of those terms is given by the condition $2 \leqslant k < m_{sp}$ which is equivalent to the condition $\sqrt{2k-1}T_c < T < \sqrt{2k+1}T_c$ in equation (12). Ansatz (13) seems therefore to be true, at least in the range of temperatures $T_c < T < \sqrt{7}T_c$, where expression (12) has been proved to be correct by Derrida. Thus, the choice (13), (14) reproduces expression (12) exactly while other possible choices for ν_p seem to lead to lower-order contributions. In [3] this assertion was proved rigorously for choices of the ν_p such as

$$\begin{aligned} \nu_1 &= n - m - r \\ \nu_m &= 1 \qquad \text{with } m > 0 \\ \nu_r &= 1 \qquad \text{with } r > m. \end{aligned} \tag{20}$$

It can also be argued that this is likely to happen in general for more complicated choices for ν_p where the insertion of more groupings is allowed. Furthermore, these arguments do not seem to be dependent on k. If this is so, the choice (13) provides the leading-order corrections to the high-temperature free energy for all $T > T_c$ and expression (12) is correct even for $k > 3$.

4. The p-spin model

All this may now be extended to the p-spin model which is defined by the Hamiltonian

$$\mathcal{H}_p(\{s\}) = -\sum_{(1\leqslant i_1<i_2<\cdots<i_p\leqslant N)} J_{i_1,i_2,\ldots,i_p} s_1\cdots s_p + h\sum_i s_i \tag{21}$$

where h is an external magnetic field and the $J_{i_1,i_2,\ldots,i_p}$ are random variables that obey the Gaussian law

$$P(J_{i_1,i_2,\ldots,i_p}) = \left[\frac{N^{p-1}}{\pi J^2 p!}\right]^{-1/2} \exp\left[-\frac{(J_{i_1,i_2,\ldots,i_p})^2 N^{p-1}}{J^2 p!}\right]. \tag{22}$$

For the sake of simplicity, from now on it will be assumed $J = 1$. If one indicates with E_i the energy relative to configuration $\{\sigma\}_i$, it is a well known result that

$$P(E_1, E_2, \ldots, E_k) \overset{p\to\infty}{\longrightarrow} \prod_{i=1}^{k} P(E_i) \qquad \text{for } |q^{i,j}| < 1, \forall i, j \tag{23}$$

where $q_{i,j}$ indicates the overlap between $(\{\sigma\}_i)$ and $(\{\sigma\}_j)$. In the limit $p \to \infty$ then, the p-spin model reduces to the REM [4,5].

The expression for $\overline{Z^n}$ is

$$\overline{Z^n} = \sum_{\{s_i^a\}} \exp\left[\frac{\beta^2 N}{4}\left(n + \sum_{a\neq b} Q_{ab}^p(s)\right)\right] \tag{24}$$

where $Q_{ab}(s) \equiv \frac{1}{N}\sum_i^N s_i^a s_i^b$.

Above $T_c = 1/2\sqrt{\ln 2}$ the model has no replica symmetry breaking. After the elimination of all the requested Lagrange multipliers, the high-temperature solution is an $n\times n$ matrix of the form

$$\begin{aligned} Q_{ab} &= q_0 \qquad \text{for } a\neq b \\ Q_{aa} &= 0. \end{aligned} \tag{25}$$

For $T < T_c$ the matrix Q_{ab} has two parameters q_0 and q_1. In the limit $p\to\infty$ the solution is $q_0 = 0$ and $q_1 = 1$ and one recovers expression (3). For simplicity h has been put equal to 0 because it will not affect the point.

Assuming now $T > T_c$, one can proceed equivalently to what has been done in the previous section and evaluate $\overline{Z^n}$ on a one-block matrix Q_{ab} of the form

$$Q_{1\text{Block}} = \begin{pmatrix} 0 & q_1 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_1 & 0 & q_1 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_1 & q_1 & 0 & q_1 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_1 & q_1 & q_1 & 0 & q_0 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & 0 & q_0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_0 & 0 & q_0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & 0 & q_0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & 0 & q_0 \\ q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & q_0 & 0 \end{pmatrix}. \tag{26}$$

$Q_{1\text{Block}}$ is an $n\times n$ matrix with an $m\times m$ block of elements q_1. The particular values $n = 9$ and $m = 4$ have been chosen in order to represent the matrix $Q_{1\text{Block}}$. It is easy to see that

$$\lim_{n\to 0}\overline{Z^n}_{1\text{Block}} = \lim_{n\to 0}\overline{Z^n}_{\text{sub}} = \lim_{n\to 0}\sum_{m=2}^{\infty}\frac{n!}{(n-m)!m!}\exp\left\{-N\left[(1-m)\ln 2 + \frac{\beta^2}{4}(m^2-m)\right]\right\} \tag{27}$$

which is the same result found in the previous section. Ansatz (13) is therefore equivalent to a matrix of the form (26). It is worth remarking that to obtain $\overline{Z^n}_{\rm 1Block}$ one has to sum over all the possible ways of inserting an $m \times m$ block in an $n \times n$ matrix.

This result will now be extended to the case $p < \infty$ in the fashion of [7]. Before that, it is appropriate to briefly show the low-temperature behaviour of the model. Below T_c, the $p < \infty$ expansion leads to the mean-field equations

$$q_0 = 0 \qquad q_1 = 1 - \frac{m\xi(m)}{(1-m)} \frac{e^{-p\beta^2 m^2/4}}{2\sqrt{p\beta^2/2}} \tag{28}$$

where

$$\xi(m) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\,(2\cosh(mz) - 2^m \cosh^m(z)). \tag{29}$$

One also has

$$T_c = \frac{1}{2\sqrt{\ln 2}} \left(1 + 2^{-(p+1)} \sqrt{\frac{\pi}{4p(\ln 2)^3}}\right). \tag{30}$$

Equation (30) for the critical temperature comes from the condition that the value of the break point in the order-parameter function [6] that maximizes the free energy is 1. This is because of the nature of the transition which, though second order in the thermodynamic sense, is discontinuous from the order-parameter point of view [7].

If we now set $T > T_c$, one can calculate $\overline{Z^n}_{\rm sub}$ with the insertion of a block, as in the $p \to \infty$ limit. Defining $T_c^\infty \equiv 1/(2\sqrt{\ln 2})$, the finite-$p$ equivalent of expression (12) can easily be obtained:

$$\overline{\ln Z} = N\left[\ln 2 + \frac{1}{4T^2}\right] - \frac{1}{2}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^2} + \frac{1}{3}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^3} + \cdots + \frac{(-1)^{k+1}}{k}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^k}$$
$$+ \frac{+2T\sqrt{\pi}\exp\left[-\frac{NT^2}{16}\left[\frac{1}{(T_c^\infty)^2} - \frac{1}{T^2}\right]^2 - N\eta + \frac{N\omega^2\beta^2}{4}\right]}{\left[\frac{T^2}{(T_c^\infty)^2} + 1 + 2\omega\right]\sqrt{N}\sin\left(\frac{\pi}{2}\left[\frac{T^2}{(T_c^\infty)^2} + 1 + 2\omega\right]\right)} + \cdots$$
$$\text{for } \sqrt{2k' - 1}(T_c^\infty) < T < \sqrt{2k' + 1}(T_c^\infty) \tag{31}$$

where

$$k' \equiv k - \omega \tag{32}$$

$$\omega \equiv 2T^2 \eta'(m) \tag{33}$$

$$\eta(m) \equiv \frac{\xi(m) e^{\beta^2 p q_i^{p-1} m^2/4}}{\sqrt{2\beta^2 p q_i^{p-1}}}. \tag{34}$$

One also has

$$m_{\rm sp} = \frac{1}{2}\left(1 + \left(\frac{T}{T_c^\infty}\right)^2\right) + \omega. \tag{35}$$

It is worthwhile noting that, by assuming $m_{\rm sp} = 1$ in equation (35), one obtains equation (30) for the critical temperature. This means that one can find T_c as the temperature at which the block inserted in the metastable matrix disappears and $Q_{\rm 1Block}$ coincides with the stable saddle-point matrix of $\overline{Z^n}_{\rm dom}$. Furthermore, equation (28) for q_1 is recovered as a saddle-point equation for $\overline{Z^n}_{\rm sub}$. In the absence of a magnetic field, it has been assumed that $q_0 = 0$.

5. The Potts model

The results obtained up till now will be extended in this section to the Potts model which is defined by the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i,j=1}^{N} J_{ij}(p\delta_{p(i)p(j)} - 1) - h_\lambda \sum_{i=1}^{N}(p\delta_{p(i),\lambda} - 1) \tag{36}$$

where $p(i) = 0, 1, \ldots, p-1$, and the J_{ij} are, as usual, random variables obeying a Gaussian distribution with variance $1/N$ and mean value J_0/N.

For every $p > 4$, the model undergoes a phase transition of the same kind as the one observed in the REM, from a replica symmetric phase to a one-symmetry-broken phase. In this work all the ferromagnetic order parameters will be neglected and the focus will be only on the glassy aspects of the model. Hence, the free energy is [3, 8]

$$\frac{n\beta F(Q)}{N} = -n\frac{\beta^2}{4}(p-1) + \frac{\beta^2}{2p^2}\sum_{\alpha<\beta}^{n}\sum_{r,s}^{p}(q_{rs}^{\alpha\beta})^2 - \ln Z(Q) \tag{37}$$

where

$$\ln Z(Q) = \ln\left\{\sum_{\{p(\alpha)\}} \exp\left[\beta^2 \sum_{\alpha<\beta}^{n} q^{\alpha\beta}_{p(\alpha)p(\beta)}\right]\right\} \tag{38}$$

and

$$q_{rs}^{\alpha\beta} = q^{\alpha\beta}(p\delta_{r,s} - 1), \qquad \text{with } 0 \leqslant |q^{\alpha\beta}| \leqslant 1. \tag{39}$$

According to conventional wisdom, one can assume that in the high-temperature phase one has $q_0 = 0$. Therefore the free energy is

$$F = -N\left[\frac{\beta}{4}(p-1) + \frac{\ln p}{\beta}\right] \tag{40}$$

and the entropy

$$S = N\left[\ln p - \frac{p-1}{4T^2}\right] \tag{41}$$

which becomes negative if $T < T_c \equiv \sqrt{(p-1)/4\ln p}$. Below T_c the solution is given by a one-symmetry-broken matrix of elements $q^{\alpha\beta}$.

To calculate the low-temperature free energy it is useful to use the p vectors $\mathbf{e}^a$ ($a = 1, \ldots, p$), defined by the relations

$$e_i^a e_i^b = p\delta_{a,b} - 1 \qquad i = 1, \ldots, p-1 \tag{42}$$

where repeated indexes are summed. If $q_0 = 0$ and $q_1 = q$, with a little algebra, one obtains the low-temperature free-energy expression:

$$\begin{aligned}\frac{\beta F}{N} = &-\frac{\beta^2}{4}(p-1) + \frac{\beta^2}{4}(p-1)(m-1)q^2 + \frac{\beta^2}{2}(p-1)q \\ &-\frac{1}{m}\ln\int\frac{\mathrm{d}\mathbf{z}\,\mathrm{e}^{(-z^2/2)}}{\sqrt{2\pi}}\left(\sum_{b=1}^{p}\exp[k\mathbf{e}^b\cdot\mathbf{z}]\right)^m\end{aligned} \tag{43}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_{p-1})$ and $k \equiv \beta\sqrt{q}$.

In the limit $p \to \infty$, the last integral can be solved exactly [3] and the solution is found to be given by the mean-field equations:

$$q_0 = 0 \qquad q_1 = 1$$

$$T_c = \frac{1}{2}\sqrt{\frac{p-1}{\ln p}}$$

$$F = \begin{cases} N\left[-\frac{\beta}{4}(p-1) - \frac{\ln p}{\beta}\right] & \text{for } T > T_c \quad (44a) \\ -E_0 = -N\sqrt{(p-1)\ln p} & \text{for } T < T_c. \quad (44b) \end{cases}$$

This solution describes a p^N-state REM with a variance proportional to $(p-1)$. The finite-size corrections for this limit are already known, so the more general case in which the finite-p corrections are included will be treated directly. One can define

$$\frac{\xi_p(m)}{p\beta\sqrt{q}} \equiv \int_{-\infty}^{\infty} \prod_{b=1}^{p} \mathrm{d}a_b \exp\left[-\tfrac{1}{2}p\left(\sum_b a_b^2\right)\right]\delta\left(\sum_{b=1}^{p} a_b\right) \times\left[\left(\sum_b^p \exp(\beta\sqrt{q}\,pma_b) - \left(\sum_b^p \exp(\beta\sqrt{q}\,pa_b)\right)^m\right]\right] \tag{45}$$

where one has

$$\xi_p(1) = 0. \tag{46}$$

Recent work has been done on the numerical estimation of the p-dimensional integral $\xi_p(m)$ [9]. Correcting the integral in the free-energy expression one obtains the mean-field equations:

$$q = 1 - \frac{1}{(\beta^3 p^2(p-1)\sqrt{q}}\frac{\xi_p(m)}{m(1-m)}\left[m^2\beta^2(p-1) + \frac{1}{q}\right]\exp[-\tfrac{1}{2}\beta^2 m^2(p-1)q]. \tag{47}$$

Assuming $\ln p \gg 1$ the last term in square brackets can be neglected:

$$q = 1 - \frac{\xi_p(m)}{(1-m)}\frac{\beta_c}{p^4\beta^2}. \tag{48}$$

Setting $q = 1 - \epsilon$, one finds an equation for the critical temperature

$$\frac{1}{T_c^2} = \frac{4}{p-1}\left(\ln p + \frac{T_c\xi_p'(1)}{p^4}\right) + \mathrm{O}(\varepsilon^2). \tag{49}$$

One can see that the corrective terms in equations (48), (49) tend to zero in the limit $p \to \infty$. The finite-size corrections are obtained by proceeding in the same way as for the p-spin. Defining

$$\eta(m) \equiv \frac{\xi_p(m)}{\beta(p-1)p^2}\exp[-\tfrac{1}{2}\beta^2 m^2(p-1)]$$
$$\omega \equiv 2T^2\eta'(m) \tag{50}$$

one gets

$$m_{\mathrm{sp}} = \frac{1}{2}\left(1 + \left(\frac{T}{T_c^{\infty}}\right)^2\right) + \omega \tag{51}$$

where $T_c^\infty \equiv \frac{1}{2}\sqrt{\frac{p-1}{\ln p}}$ and finally

$$\overline{\ln Z} = N\left[\ln p + \frac{p-1}{4T^2}\right] - \frac{1}{2}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^2} + \frac{1}{3}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^3} + \cdots + \frac{(-1)^{k+1}}{k}\overline{\left[\frac{Z}{\overline{Z}} - 1\right]^k}$$
$$+ \frac{2T\sqrt{\pi}\exp\left[-\frac{NT^2}{16}\left[\frac{1}{(T_c^\infty)^2} - \frac{1}{T^2}\right]^2 - N\eta + \frac{N\omega^2\beta^2}{4}\right]}{\left[\frac{T^2}{(T_c^\infty)^2} + 1 + 2\omega\right]\sqrt{N}\sin\left(\frac{\pi}{2}\left[\frac{T^2}{(T_c^\infty)^2} + 1 + 2\omega\right]\right)} + \cdots$$
$$\text{for } \sqrt{2k'-1}(T_c^\infty) < T < \sqrt{2k'+1}(T_c^\infty) \tag{52}$$

where

$$k' \equiv k - \omega. \tag{53}$$

In principle, all this is equivalent to what has been done for the Ising spin glass with p-spin interactions and all the considerations made at the end of section 4 can be repeated here.

6. Conclusion

For the high-temperature phase of the REM, the partition function was evaluated on a metastable point that was introduced in order to account for the probability that a group of m replicas freezes in a phase which resembles the low-temperature one. In this way the finite-size corrections to the free energy were calculated. The result was checked with the one obtained by Derrida without the use of replicas. The two approaches are totally independent. Derrida hypothesizes that his result could be valid for every $T > T_c$ but was only able to prove it in the range of temperatures $T_c < T < \sqrt{7}T_c$. The results obtained in this work coincide with Derrida's, in particular they coincide for $T_c < T < \sqrt{7}T_c$. The reliability of this method does not seem to depend on the temperature, provided that $T > T_c$. Therefore, the equivalence of the two results in the range where formula (12) can be proved to be true seems to indicate the reliability of the result for all temperatures. In extending this ansatz to the p-spin and Potts models it was possible to identify a one-block matrix as the metastable point. Mean-field equations give, for the elements of the block, the same value as the low-temperature mean overlap. Furthermore, a $p < \infty$ expansion was performed for these models in order to extend the results to finite-p, p-spin and Potts models.

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